Pergamon

J. Appl. Maths Mechs, Vol. 57, No. 5, pp. 857–860, 1993

0021–8928/93 \$24.00+0.00

0021–8928(93)E0041-T

A SUFFICIENT CONDITION OF INSTABILITY FOR DYNAMICAL SYSTEMS WITH DISTRIBUTED PARAMETERS DEPENDING ON A SINGLE SPATIAL VARIABLE†

P. A. YAKUBENKO

Moscow

(*Received* 2 *March* 1993)

An effective sufficient condition for the instability of a certain class of one-dimensional weakly non-uniform dynamical systems of arbitrary nature is formulated. On the basis of the proposed condition the instability of an infinitely extended weakly non-uniform elastic pipe filled with moving fluid is considered.

1. CONSIDER a system whose parameters depend on a single spatial variable x and do not depend on time t. Suppose this dependence is described by some typical length-scale L. There is a certain fundamental state of the system $\Psi(x)$ whose stability with respect to small perturbations with characteristic wavelength λ it is required to investigate. It is assumed that the parameters of the system under investigation change fairly slowly, i.e. $\varepsilon=\lambda/L\ll 1$. We introduce the slow variable $X=\varepsilon x$. Let the behaviour of small perturbations $\psi(x, X, t)$ be described by the equation

$$D\left[-i\frac{\partial}{\partial x},\ i\frac{\partial}{\partial t},\ \mathbf{R}(X),\ \varepsilon\frac{d\mathbf{R}}{dX}(X),\ldots\right]\psi(x,X,t)=0 \tag{1.1}$$

where D is a linear operator and $\mathbf{R}(x)$ is the system parameter vector. As usual, a solution of Eq. (1.1) is sought in the form $\phi(x, X)e^{-i\omega t}$, where $\omega\in C$ is an unknown eigenfrequency and $\phi(x,X)$ is an eigenfunction satisfying appropriate boundary conditions.

The fundamental state $\Psi(x)$ is considered to be stable [1] if for all possible eigenfrequencies $\operatorname{Im}\omega<0$, and unstable if at least one eigenfrequency ω_* exists such that

$$\operatorname{Im}\omega_*>0 \tag{1.2}$$

"Freezing" X in Eq. (1.1) and taking $\phi(x, X)=Ae^{ikx}$ we obtain the local dispersion relation

$$D(k,\omega,\mathbf{R}(X),0,\ldots)=0 \tag{1.3}$$

One can argue that Eq. (1.3) corresponds to some fictitious uniform system with parameters corresponding to the given X. Such a uniform system can be investigated for the presence of absolute instability [2, 3]. In [4, 5] the following approach is proposed for finding the eigenfrequencies ω_*. A solution $k_0(X)$ of equation $\partial\omega(k, X)/\partial k=0$ (indicating the presence

†*Prikl. Mat. Mekh.* Vol. 57, No. 5, pp. 100–104, 1993.

of absolute instability) is chosen in some appropriate manner [3] for a fixed value of X. Here $\omega(k, X)$ is a solution of Eq. (1.3). The function $\omega_0(X) \equiv \omega(k_0(X), X)$ is also defined, and we consider the analytic continuation of this function in the complex X plane. If the closest singularity of the function $\omega_0(X)$ to the $\mathrm{Re}X$ axis is a saddle point X_S, i.e. a point at which the condition $\partial\omega_0(X_S)/\partial X = 0$ is satisfied, then an eigenfrequency $\omega_* = \omega_S + O(\varepsilon)$, exists [4, 5] where $\omega_S = \omega_0(X_S)$. In particular, if $\mathrm{Im}\,\omega_S > 0$, the system under consideration is unstable in the sense of definition (1.2).

2. The criterion given above can serve as a sufficient condition for instability for a wide range of dynamical systems. However, in general it requires an investigation of the behaviour of the function $\omega_0(X)$ in the complex X plane, which can present serious difficulties when the form of this function is complicated. In this paper, we present instability criteria for a class of systems where a conclusion of instability can be obtained as a result of investigating system parameters for purely real values of X. We have the following theorem.

Theorem. Suppose a dynamical system has the following properties:

1. All system parameters $R_1(X), \ldots, R_n(X)$ are analytic functions in some domain of the complex X plane that includes the real X axis.

2. $\forall X \in \mathbf{R}$: $\mathrm{Im}\,R_j(X) = 0, \quad j = 1,\ldots,n$

3. $\exists X_0 \in \mathbf{R}$: $\dfrac{\partial R_j}{\partial \mathrm{Re}\,X}(X_0) = 0, \quad j = 1,\ldots,n$

4. The uniform system with parameters $R_1(X_0), \ldots, R_n(X_0)$ is absolutely unstable.

Then this system is unstable in the sense of (1.2).

Proof. In the given system the function $\omega_0(X)$ can be represented in the form $\omega_0(X) = \Omega_0(R_1(X), \ldots, R_n(X))$, whence

$$\frac{\partial\omega_0}{\partial X}(X) = \sum_{j=1}^{n} \frac{\partial\Omega_0}{\partial R_j} \frac{\partial R_j}{\partial X}(X) \tag{2.1}$$

From assumptions 1–3 it follows that

$$\frac{\partial R_j}{\partial X}(X_0) = 0, \quad j = 1,\ldots,n \tag{2.2}$$

Then from (2.1) and (2.2) we find that $\partial\omega_0(X_0)/\partial X = 0$). This means that X_0 is a saddle point of the function $\omega_0(X)$. By assumption 4 we find that in accordance with the definition of $\omega_0(X)$, $\mathrm{Im}\,\omega_0(X_0) > 0$. This also means that the system considered is unstable in the sense of definition (1.2).

3. We will illustrate the application of this theorem, together with the criteria of [4, 5], to the problem of the behaviour of small bending oscillations of an elastic tube that is infinitely extended in both directions and filled with moving fluid. Suppose that the parameters of the system under consideration depend only slightly on the spatial variable x and do not depend on the time t. It is assumed that the tube deformation can be reduced to the bending of its axis, with the transverse sections of the tube remaining unchanged. The equation for the deflection of the tube $\xi(x, t)$ has the form [7]

$$\rho_1(x)\frac{\partial^2\xi}{\partial t^2} + \rho_2(x)\left(\frac{\partial}{\partial t} + \upsilon(x)\frac{\partial}{\partial x}\right)^2 \xi = -\frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2\xi}{\partial x^2}\right)$$

where $\rho_1(x)$ and $\rho_2(x)$ are the mass per unit length of the tube and fluid, respectively, $D(x)$ is the bending rigidity of the tube, and $\upsilon(x)$ is the velocity of the fluid.

For each fixed value of $X=\varepsilon x$ one obtains the dispersion relation

$$D(X)k^4-\upsilon^2(X)k^2+2\rho_2(X)\upsilon(X)\omega k-\omega^2(\rho_1(X)+\rho_2(X))=0$$

For fixed X this equation corresponds to a tube with constant parameters. It was established in [8] that a uniform system of this type is linearly unstable for all values of the parameters. It was also found that the instability is absolute if

$$\rho_2(X)>8\rho_1(X) \tag{3.1}$$

We will impose an additional restriction on the form of the system parameters. We shall assume that the tube is circular, with external radius $R(X)$ and internal radius $r(X)$, and consists of a uniform material of density ρ_T. We will assume that the fluid is ideal, incompressible, and of density ρ_F. Then the system parameters have the form

$$\rho_1(X)=\pi\rho_T(R^2(X)-r^2(X)),\quad \rho_2(X)=\pi\rho_F r^2(X)$$

$$D(X)=E\frac{\pi}{4}(R^4(X)-r^4(X)),\quad \upsilon(X)=Q/(\pi\rho_F r^2(X))$$

where R is Young's modulus of the tube and Q is the mass flow rate of the fluid through the tube. Thus all system parameters are functions of just R and r.

We will use the theorem proved in Sec. 2 to investigate the circular tube described above. Suppose that on the real X axis there is some region D in which the absolute instability condition (3.1) for a uniform system is satisfied. Suppose also that at some point in the region D the functions $R(X)$ and $r(X)$ simultaneously reach extremal values. Then from the theorem one can conclude that the tube is unstable.

As specific examples of the above situation we have circular tubes with constant internal or external radii, and also tubes with walls of constant thickness. If such tubes have sections satisfying condition (3.1), then they are unstable.

We will also consider a case of tube instability in which to apply the instability conditions of [4, 5] one must investigate the behaviour of the function $\omega_0(X)$ in the complex X plane. Suppose that in dimensionless variables the functions $R(X)$ and $r(X)$ have the following form

$$R(X)=a_1+b/\left((X+c)^2+d\right)$$

$$r(X)=a_2-b/\left((X-c)^2+d\right)$$

where a_1, a_2, b, c and d are real constants. For $c\neq 0$ the parameters of the system under consideration obviously do not satisfy the conditions of the theorem given in Section 2.

In the problem under consideration the function $\omega_0(X)$ has the form

$$\omega_0(X)=\frac{\upsilon^2(X)}{16}\left(\frac{D(X)}{\rho_1(X)+\rho_2(X)}(8\beta(X)-6\beta^2(X)+2\alpha(X))\right)^{1/2}(3\beta(X)+\alpha(X))$$

$$\beta(X)=\rho_2(X)/(\rho_1(X)+\rho_2(X)),\quad \alpha(X)=i(8\beta(X)-9\beta^2(X))^{1/2}$$

Because the form obtained for the function $\omega_0(X)$ is somewhat complex, we investigated it numerically. The shape of the function $\omega_0(X)$ in the neighbourhood of the real X axis was investigated over a wide range of parameter values. For all parameter values considered it was found that if the absolute

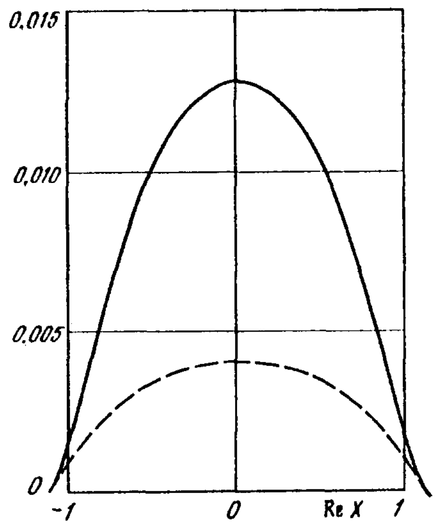

FIG. 1.

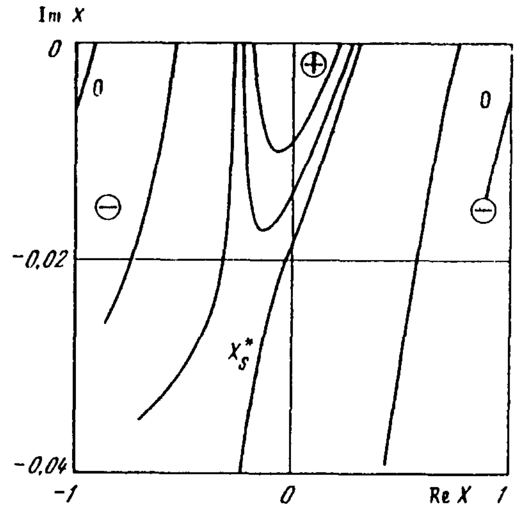

FIG. 2.

instability condition (3.1) was satisfied on some section of the real X axis, then near to this section in the complex X plane a saddle point X_S of the function $\omega_0(X)$ exists, with $\text{Im}\,\omega_0(X_S)>0$. Thus the system under consideration is unstable according to the criteria of [4, 5].

Figure 1 shows graphs of the function $\text{Im}\,\omega_0(X)$ (the solid curve) and $\rho_1(X)/\rho_2(X)-1/8$ (the dashed curve) for real X with $a_1=1.995$, $a_2=2.005$, $b=0.015$, $c=1$, $d=4$. It can be seen from these graphs that the absolute instability condition for the uniform system corresponding to a given X is identical with (3.1). Figure 2 shows the curves $\text{Im}\,\omega_0(X)=\text{const}$ in the complex X plane, obtained by numerically investigating $\omega_0(X)$. The plus (minus) signs indicate regions of growing (decreasing) $\text{Im}\,\omega_0$. The lines $\text{Im}\,\omega_0=0$ are marked with a zero. At the saddle point located at $X_S=(-0.25;\ -0.026)$ we have $\text{Im}\,\omega_0(X_S)=0.0121>0$, which indicates the instability of the system.

We note in conclusion that, as has been shown in [9], the instability criteria formulated in this paper, like the criteria in [4, 5], are only sufficient for instability, and the non-satisfaction of these criteria does not imply that the system investigated is stable.

This work was supported by the Russian Fund for Fundamental Research (93-013-17355).

REFERENCES

1. LIN C. C., *Theory of Hydrodynamic Stability.* Izd. Inostr. Lit., Moscow, 1958.
2. BERS A. and BRIGGS R. J., Criteria for determining absolute instability and distinguishing between amplifying and evanescent waves. *Bull. Am. Phys. Soc.* **9**, 3, 304–305, 1964.
3. LIFSHITS Ye. M. and PITAYEVSKII L. P., *Theoretical Physics*, Vol. 10: *Physical Kinetics*. Nauka, Moscow, 1979.
4. CHOMAZ J. M., HUERRE P. and REDEKOPP L. G., A frequency selection criterion in spatially developing flows. *Stud. Appl. Math.* **84**, 2, 119–144, 1991.
5. HUERRE P. and MONKEWITZ P. A., Local and global instabilities in spatially developing flows. *Ann. Rev. Fluid Mech.* **22**, 473–537, Ann. Revs. Inc., Palo Alto, California, 1990.
6. KULIKOVSKII A. G., On the stability of uniform states. *Prikl. Mat. Mekh.* **30**, 1, 148–153, 1966.
7. BREKHOVSKIKH L. M. and GONCHAROV V. V., *Introduction to the Mechanics of Continuous Media.* Nauka, Moscow, 1982.
8. KULIKOVSKII A. G. and SHIKINA I. S., On the elastic oscillations of a long tube filled with a moving fluid. *Izv. Akad. Nauk ArmSSR*, **41**, 1 31–39, 1988.
9. KULIKOVSKII A. G., On the loss of stability of weakly non-uniform flows in extended regions. Formation of transverse oscillations of a tube conveying a fluid. *Prikl. Mat. Mekh.* **57**, 5, 93–99, 1993.

Translated by R.L.Z.